Example 3.2.1. Discuss the continuity of the function $f : [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \dfrac{x-1}{x} & \text{if } x \in (0,1], \\ 0 & \text{if } x = 0. \end{cases}$$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} \frac{x-1}{x} = 0$

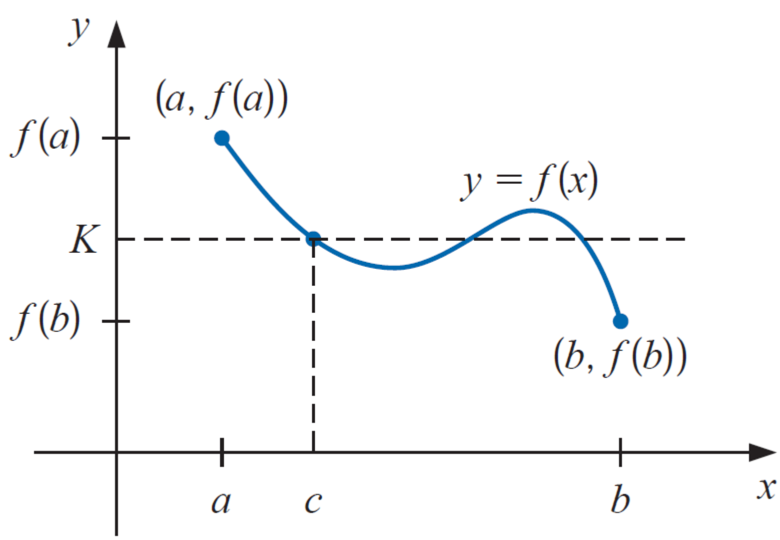
$f(1) = \frac{1-1}{1} = 0$

Solution. $f(x)$ is continuous on $(0,1)$. $f(x)$ is also continuous at $x=1$, but $\lim_{x\to 0^+} f(x)$ does not exists. So f is not continuous at $x=0$. ∎

$f(0) = 0 \quad \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{x-1}{x} = -\infty \neq f(0)$

Theorem 3.2.1 (Intermediate Value Theorem or Intermediate Value Property). *Suppose f is a continuous function on $[a,b]$ and K is a number between $f(a)$ and $f(b)$. Then there exist a number c, between a and b, such that $f(c) = K$.*

Geometrically, the Intermediate Value Theorem says that any horizontal line $y = y_0$ crossing the y-axis between the numbers $f(a)$ and $f(b)$ will cross the curve $y = f(x)$ at least once over the interval $[a,b]$.



Application: Root Finding

If $f(x)$ is continuous on $[a,b]$, $f(a)$ and $f(b)$ change sign, then, there exists at least one root of the function, that is, exists at least one $c \in (a,b)$, such that $f(c) = 0$.

Example 3.2.2. Show that $f(x) = x^5 - x + 1$ has a root.

in general, when x is very large $f(x) \sim x^5 \to \infty$

$(\lim_{x\to+\infty} f(x) = +\infty)$

$\lim_{x\to-\infty} f(x) = -\infty$

$\to \exists\, c \in \mathbb{R}$ s.t $f(c) = 0$

Solution. Aim: find a, b, such that $f(a)$, $f(b)$ change sign. Since

$$f(-2) = -29, \quad f(0) = 1,$$

and f is continuous on $[-2, 0]$. By Intermediate value theorem, there exists $c \in (-2, 0)$, such that $f(c) = 0$.

■

Remark. Although we don't know how to find the root, we know a root exists.

Example 3.2.3. 1. All odd functions have a root.

nonzero

$f(x) = -f(-x)$

$f(x_0) \begin{smallmatrix} > 0 \\ < 0 \end{smallmatrix}$ or.

$\Downarrow$

$f(-x_0) \begin{smallmatrix} < 0 \\ > 0 \end{smallmatrix}$

2. All polynomials of odd degrees have a root.

Exercise 3.2.1. Show that $2^x = \dfrac{1}{x^2}$ has a solution.

$f(x) = 2^x - \frac{1}{x^2}$ $\quad f(1) = 1 \quad \lim_{x \to 0^+} f(x) = 2^0 - \infty = -\infty$

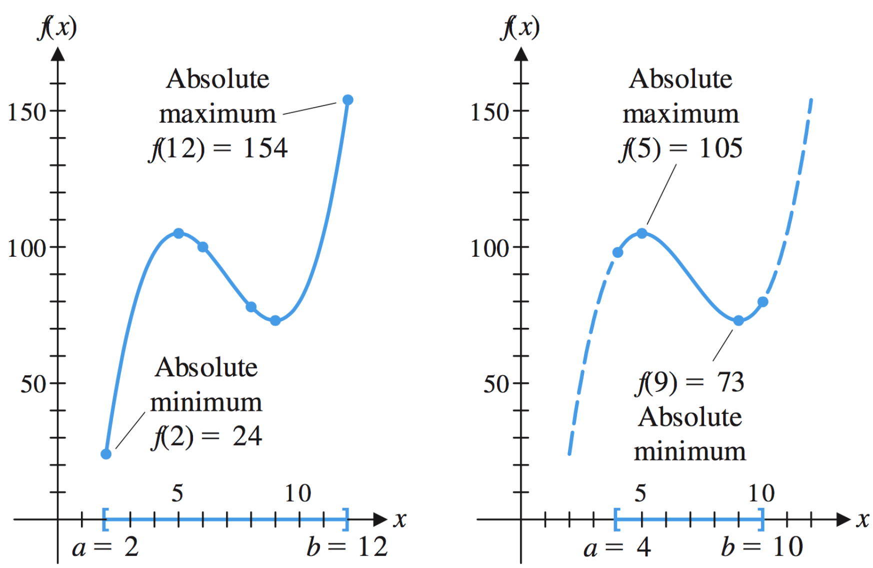
$\exists\, c,\ 0 < c < 1$ s.t. $f(c) = 0$ ∵ opposite sign.

Theorem 3.2.2 (Extreme Value Theorem). *If $f(x)$ is continuous on $[a, b]$, then f must attain an absolute maximum and absolute minimum, that is, there exist c, d in $[a, b]$ such that*

$$f(c) \leq f(x) \leq f(d),$$

for all $x \in [a, b]$.

Example 3.2.4. Absolute extreme for $f(x) = x^3 - 21x^2 + 135x - 170$ for various closed intervals.

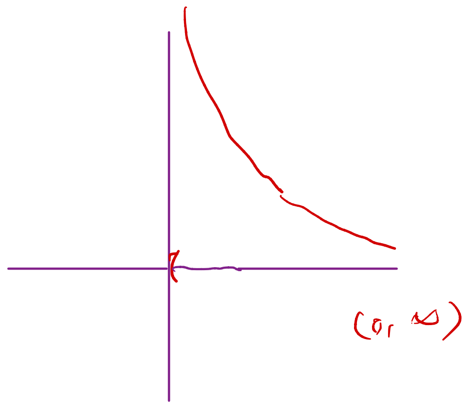


(A) $[a, b] = [2, 12]$ (B) $[a, b] = [4, 10]$

Exercise 3.2.2 (Hard!). Derive the extreme value theorem from the intermediate value theorem.

Remark. Caveat: The intermediate value theorem and the extreme value theorem only work on *finite* and *closed* intervals! E.g. Consider the previous example on $\mathbb{R}$, and $\dfrac{1}{x}$ on $\mathbb{R}^+$ or on $(0, 1)$.

E.g,



Question: How to find the absolute maximum and minimum?

Ans: (for "good" functions) Differentiation!

Exercise 3.2.2 (Hard!). Derive the extreme value theorem from the intermediate value theorem.

Remark. Caveat: The intermediate value theorem and the extreme value theorem only work on *finite* and *closed* intervals! E.g. Consider the previous example on $\mathbb{R}$, and $\dfrac{1}{x}$ on $\mathbb{R}^+$ or on $(0, 1)$.

Question: How to find the absolute maximum and minimum?

Ans: (for "good" functions) Differentiation!

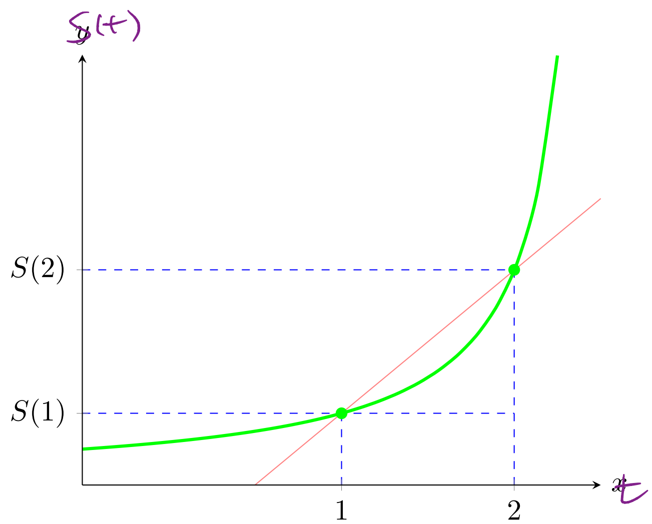
MATH1520 University Mathematics for Applications **Fall 2021**

Chapter 4: Differentiation I

Learning Objectives:
(1) Define the derivatives, and study its basic properties.
(2) Study the relationship between differentiability and continuity.
(3) Use the constant multiple rule, sum rule, power rule, product rule, quotient rule and chain rule to find derivatives.
(4) Explore logarithmic differentiation.

4.1 Motivation & Definition

Motivation from physics: velocity Suppose an object is moving along x-axis from the origin to right. Let $S = S(t)$ be the position of the object at time t. What is the average velocity of this object from $t = 1$ to $t = 2$?



$$\begin{aligned}
\text{Average velocity from } t=1 \text{ to } t=2 &= \frac{\text{Change of position}}{\text{Change of time}} \\
&= \frac{\Delta S}{\Delta t} \\
&= \frac{S(2)-S(1)}{2-1} \\
&= \text{slope of secant line passing through } (1,S(1)) \text{ and } (2,S(2))
\end{aligned}$$

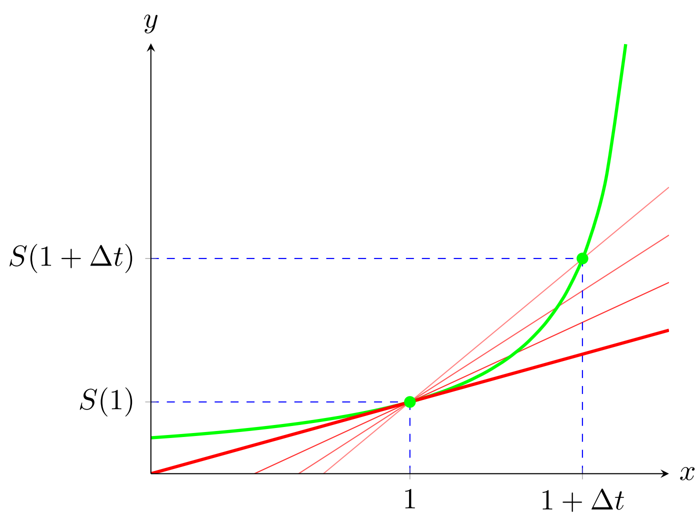
Question: What is the instantaneous velocity at $t = 1$?

Idea: Average velocity from $t = 1$ to $t = 1 + \Delta t$ is $\dfrac{S(1+\Delta t)-S(1)}{\Delta t}$, where Δt is small.

Let $\Delta t \to 0$, the instantaneous velocity at $t = 1$ is defined to be

$$S'(1) = \lim_{\Delta t \to 0} \frac{S(1+\Delta t)-S(1)}{\Delta t},$$

which is called the **derivative** of S at $t = 1$. $S'(1)$ describes the rate of change of $S(t)$ at $t = 1$.



Remark. Terminology: The term "velocity" takes the direction of motion into account; it can be positive or negative. The term "speed" only takes into account the rate of change, disregarding the direction. It is the absolute value of the velocity.

Definition 4.1.1. The **derivative** of $f(x)$ is the function

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}. \tag{4.1}$$

The process of computing the derivative is called **differentiation**, and we say that $f(x)$ is **differentiable** at $x = x_0$ if $f'(x_0)$ exists; that is, $\lim\limits_{\Delta x \to 0} \dfrac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$ exists.

Remark. 1. By definition, if $f(x_0)$ is not well-defined, we cannot define $f'(x_0)$. So $f(x)$ must not be differentiable at $x = x_0$.

2. Another equivalent formula:

$$\frac{df}{dx}(x_0) = f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

3.
$$\frac{df}{dx} \Longleftarrow \frac{\Delta f}{\Delta x} = \frac{f(x) - f(x_0)}{x - x_0}$$

is called **difference quotient.** Δ difference $\to 0 \rightsquigarrow d$

4. $f'(x_0)$ describes the rate of change of $f(x)$ at $x = x_0$.

5. When we say that we use **the first principle** to find derivatives, we mean that we use the definition (4.1) to find the derivative. However, later we will learn faster techniques to find derivatives.

Geometrical interpretation of differentiation: $f'(x_0)$ is the slope of tangent line to the curve of $f(x)$ at $x = x_0$.

Example 4.1.1. Let $f(x) = x^2$. Then (i) prove that $f(x)$ is differentiable at $x = 1$; (ii) find $f'(1)$ and the equation of the tangent line to the graph of f at $x = 1$.

Solution. (i) By the definition, at $x = 1$

$$f'(1) = \lim_{\Delta x \to 0} \frac{f(1 + \Delta x) - f(1)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(1 + \Delta x)^2 - 1^2}{\Delta x}$$
$$= \lim_{\Delta x \to 0} (2 + \Delta x)$$
$$= 2,$$

$1 + 2\Delta x + (\Delta x)^2 - 1 = \Delta x(2 + \Delta x)$

So, f is differentiable at 1, and $f'(1) = 2$.

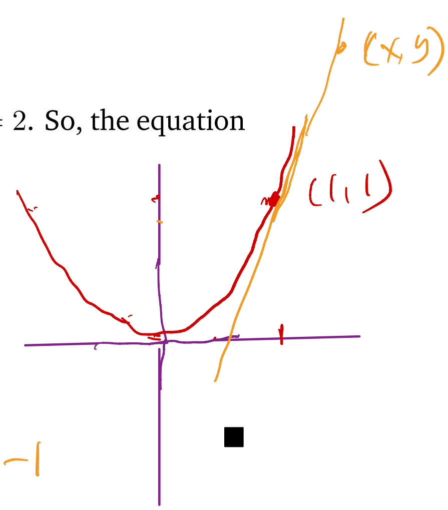
(ii) The tangent line passes through $(1, f(1)) = (1, 1)$ with slope $f'(1) = 2$. So, the equation of the tangent line is

$$\frac{y - f(1)}{x - (1)} = 2. = f'(1)$$

Thus

$$\frac{y-1}{x-1} = 2 \qquad y = 2x - 1.$$

$y - 1 = 2(x - 1) \rightsquigarrow y = 2x - 1$



■

Definition 4.1.2. If $f(x) : A \to \mathbb{R}$ is differentiable at every point $x \in A$, then $f(x)$ is said to be a differentiable function in A, and the derivative function $f'(x) : A \to \mathbb{R}$ is well-defined.

Example 4.1.2. Let $f(x) = x^2$. Prove that $f(x)$ is differentiable on $\mathbb{R}$, and find $f'(x)$.

$(x+\Delta x+x)(\cancel{x}+\Delta x-\cancel{x})$ use $a^2-b^2=(a+b)(a-b)$

Solution. For any $x \in \mathbb{R}$,

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - x^2}{\Delta x} = \lim_{\Delta x \to 0} (2x + \Delta x) = 2x.$$

So, f is differentiable at x, and $f'(x) = 2x$. ∎

Notation: For $y = f(x) = x^2$,

$$f'(x) = \frac{dy}{dx} = \frac{df}{dx} = 2x; \quad f'(4) = \left.\frac{dy}{dx}\right|_{x=4} = \left.\frac{df}{dx}\right|_{x=4} = 2 \cdot 4 = 8.$$

Question Where does the minimum of x^2 occur? (Hint: what is the slope of the tangent line at the minimum?)

Example 4.1.3. Let $f(x) = \dfrac{x+1}{x-1}$. Using the definition of derivatives, compute $f'(x)$ for $x \neq 1$.

Solution.

$$f(x+\Delta x) - f(x) = \frac{x+\Delta x+1}{x+\Delta x-1} - \frac{x+1}{x-1}$$
$$= \frac{(x-1)(x+\Delta x+1) - (x+1)(x+\Delta x-1)}{(x-1)(x+\Delta x-1)}$$
$$= \frac{-2\Delta x}{(x-1)(x+\Delta x-1)}.$$

Therefore

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{-2}{(x-1)(x+\Delta x-1)}$$
$$= \frac{\lim_{\Delta x \to 0} (-2)}{\lim_{\Delta x \to 0} (x-1)(x+\Delta x-1)} = \frac{-2}{(x-1)^2}.$$

∎

$$= (x-1) \lim_{\Delta x \to 0} (x+\Delta x-1)$$

$$= (x-1)(x+0-1)$$

Example 4.1.4. Find the derivative of $f(x) = \sqrt{x}$ for $x > 0$.

$a^2 - b^2 = (a+b)(a-b)$

Solution.

$$\begin{aligned}
\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} &= \lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{(\sqrt{x+\Delta x} - \sqrt{x})(\sqrt{x+\Delta x} + \sqrt{x})}{\Delta x(\sqrt{x+\Delta x} + \sqrt{x})} \\
&= \lim_{\Delta x \to 0} \frac{1}{\sqrt{x+\Delta x} + \sqrt{x}} \\
&= \frac{1}{2\sqrt{x}}.
\end{aligned}$$

$= (x + \Delta x) - x$

$= \Delta x$

So, $\left(x^{\frac{1}{2}}\right)' = \frac{1}{2}x^{-\frac{1}{2}}, x > 0.$ ■

Example 4.1.5. Find the derivative of $f(x) = \sqrt[3]{x}$.
Hint: $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$.

$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

Solution. For any $x \neq 0$,

$$\begin{aligned}
\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} &= \lim_{\Delta x \to 0} \frac{\sqrt[3]{x+\Delta x} - \sqrt[3]{x}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{(\sqrt[3]{x+\Delta x} - \sqrt[3]{x})((\sqrt[3]{x+\Delta x})^2 + \sqrt[3]{x+\Delta x} \cdot \sqrt[3]{x} + (\sqrt[3]{x})^2)}{\Delta x((\sqrt[3]{x+\Delta x})^2 + \sqrt[3]{x+\Delta x} \cdot \sqrt[3]{x} + (\sqrt[3]{x})^2)} \\
&= \lim_{\cancel{h} \to 0} \frac{x + \Delta x - x}{\Delta x((\sqrt[3]{x+\Delta x})^2 + \sqrt[3]{x+\Delta x} \cdot \sqrt[3]{x} + (\sqrt[3]{x})^2)} \\
&= \lim_{\Delta x \to 0} \frac{1}{(\sqrt[3]{x+\Delta x})^2 + \sqrt[3]{x+\Delta x} \cdot \sqrt[3]{x} + (\sqrt[3]{x})^2} \\
&= \frac{1}{3(\sqrt[3]{x})^2} = \frac{1}{3}x^{-\frac{2}{3}}.
\end{aligned}$$

Δx

For $x = 0$,

$$\lim_{\Delta x \to 0} \frac{f(0+\Delta x) - f(0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sqrt[3]{\Delta x} - \sqrt[3]{0}}{\Delta x} = \lim_{\Delta x \to 0} \frac{1}{(\Delta x)^{\frac{2}{3}}} \quad \text{does not exist.}$$

So,

$$(x^{1/3})' = \begin{cases} \frac{1}{3}x^{-\frac{2}{3}}, & x \neq 0 \\ \text{Not exist at } x = 0 \text{, i.e. } x^{\frac{1}{3}} \text{ not differentiable at } 0 \end{cases}$$

■

Example 4.1.6. Discuss the differentiability of $f(x) = |x|$. $= \begin{cases} x & \text{when } x \geq 0 \\ -x & \text{when } x < 0 \end{cases}$

Solution. For $x_0 > 0$,

$$\lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(x_0 + \Delta x) - x_0}{\Delta x} = 1.$$

For $x_0 < 0$,

$$\lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \cancel{=} \lim_{\Delta x \to 0} \frac{-(x_0 + \Delta x) - (-x_0)}{\Delta x} = -1.$$

For $x_0 = 0$.

$$\lim_{\Delta x \to 0^+} \frac{f(0 + \Delta x) - f(0)}{\Delta x} = \lim_{\Delta x \to 0^+} \frac{\Delta x}{\Delta x} = 1.$$
$$\lim_{\Delta x \to 0^-} \frac{f(0 + \Delta x) - f(0)}{\Delta x} = \lim_{\Delta x \to 0^-} \frac{-\Delta x}{\Delta x} = -1.$$

$1 \neq -1$, so f is not differentiable at $x = 0$. So,

$$\lim_{\Delta x \to 0} \frac{f(0+\Delta x) - f(0)}{\Delta x}$$

DNE

$$(|x|)' = \begin{cases} 1 & \text{if } x > 0, \\ \text{undefined} & \text{if } x = 0. \\ -1 & \text{if } x < 0, \end{cases}$$

■

4.2 Properties of derivatives

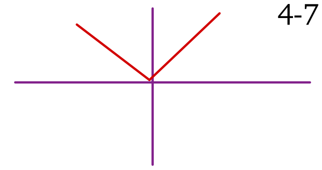
4.2.1 Differentiation and Continuity

Proposition 1. $f(x)$ is differentiable at $x = x_0 \implies f(x)$ is continuous at $x = x_0$.

Proof. Suppose $f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$ exists, then

$$\lim_{x \to x_0} f(x) - f(x_0) = \lim_{x \to x_0} (f(x) - f(x_0)) = \lim_{x \to x_0} \left(\frac{f(x) - f(x_0)}{x - x_0} \cdot (x - x_0) \right)$$
$$= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \cdot \lim_{x \to x_0} (x - x_0)$$
$$= f'(x_0) \cdot 0 = 0.$$

So, $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} (f(x) - f(x_0)) + \lim_{x \to x_0} f(x_0) = 0 + f(x_0) = f(x_0)$, that is, $f(x)$ is continuous at x_0. □



The converse is not true. For example, let $f(x) = |x|$. It is not differentiable at $x = 0$ but is continuous at $x = 0$.

Exercise 4.2.1. Let

$$f(x) = \begin{cases} x^2 - 1, & \text{if } x \geq 1 \\ 1 - x, & \text{if } x < 1 \end{cases}$$

(a) Show that $f(x)$ is continuous at $x = 1$.

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x^2-1) = 0$ $\quad \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (1-x) = 0$

(b) Show that $f(x)$ is differentiable everywhere except $x = 1$, and

so $\lim_{x\to 1} f(x) = 0 = f(0)$

$$f'(x) = \begin{cases} 2x, & \text{if } x > 1 \\ \text{undefined}, & \text{if } x = 1 \\ -1, & \text{if } x < 1 \end{cases}$$

$f'(1) = \lim_{\Delta x\to 0} \frac{f(1+\Delta x) - f(1)}{\Delta x}$

$\lim_{\Delta x\to 0^+} \frac{f(1+\Delta x) - f(1)}{\Delta x} = \lim_{\Delta x\to 0^+} \frac{(1+\Delta x)^2 - 1 - (1^2-1)}{\Delta x} = \lim_{\Delta x\to 0^+} \frac{2\Delta x + \Delta x^2}{\Delta x} = \lim_{\Delta x\to 0^+} (2+\Delta x) = 2$

$\lim_{\Delta x\to 0^-} \frac{f(1+\Delta x) - f(1)}{\Delta x} = \lim_{\Delta x\to 0^-} \frac{1-(x+\Delta x)+f(1)}{\Delta x} = \lim_{\Delta x\to 0^-} \frac{-\Delta x}{\Delta x} = -1$

4.2.2 Differentiation and Arithmetic Operations

Theorem 2. *Let $f(x)$ and $g(x)$ be differentiable functions. Then*

(1) *Sum rule:* $\quad (f + g)'(x) = f'(x) + g'(x).$

(2) *Difference rule:* $\quad (f - g)'(x) = f'(x) - g'(x).$

(Leibniz rule) (3) *Product rule:* $\quad \boxed{(fg)'(x) = f'(x)g(x) + f(x)g'(x).}$

(4) *Quotient rule:* $\quad \boxed{\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}.}$ ← derived from the Leibniz rule + "chain rule"

Proof. (1)

$$\begin{aligned}
(f+g)'(x) &= \lim_{\Delta x\to 0} \frac{(f+g)(x+\Delta x) - (f+g)(x))}{\Delta x} \\
&= \lim_{\Delta x\to 0} \frac{f(x+\Delta x) + g(x+\Delta x) - (f(x) + g(x))}{\Delta x} \\
&= \lim_{\Delta x\to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} + \lim_{\Delta x\to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x} \\
&= f'(x) + g'(x).
\end{aligned}$$

(3)

$$
\begin{aligned}
(fg)'(x) &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x+\Delta x)g(x) + f(x+\Delta x)g(x) - f(x)g(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x+\Delta x)g(x)}{\Delta x} + \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x) - f(x)g(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left(f(x+\Delta x) \cdot \frac{g(x+\Delta x) - g(x)}{\Delta x} \right) + \lim_{\Delta x \to 0} \left(\frac{f(x+\Delta x) - f(x)}{\Delta x} \cdot g(x) \right) \\
&= \lim_{\Delta x \to 0} f(x+\Delta x) \cdot \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x} + \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \cdot \lim_{\Delta x \to 0} g(x) \\
&= f(x)g'(x) + f'(x)g(x).
\end{aligned}
$$

Remark. Here we used:

$$g(x) \text{ is differentiable at } x \quad \Rightarrow \quad g(x) \text{ is continuous at } x$$

so, $\lim_{\Delta x \to 0} f(x+\Delta x) = f(x)$.

□

Exercise 4.2.2. Prove other rules using the first principle.

Remark. 1. The product rule is more commonly referred to as the *Leibniz rule*.
Caveat: $(f \cdot g)' \neq f' \cdot g'$!
2. The quotient rule (4) can be derived from the Leibniz rule together with the chain rule (Section 4.3).

4.2.3 Derivatives of Elementary Functions

Theorem 3 (Constant functions).

$$\boxed{f(x) = k \quad \Rightarrow \quad f'(x) = 0}$$

Proof.

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{k-k}{\Delta x} = 0.$$

□

As a consequence, we have

$$\boxed{(kf(x))' = (k)'f(x) + kf'(x) = kf'(x),} \quad \text{for any constant } k.$$

Remark. It can also be proved by the first principle.

Eg, $f(x) = x$

$f'(x) = \lim_{\Delta x \to 0} \frac{x + \Delta x - x}{\Delta x}$

$= 1$

$(x^2)' = (x \cdot x)'$

$= x' \cdot x + x \cdot x'$

$= 2x$

$(x^3)' = (x \cdot x^2)'$

$= \dots$

Theorem 4 (The Power Rule)**.**

$$\boxed{(x^a)' = ax^{a-1},} \quad \textit{whenever it is well-defined, } a \in \mathbb{R}.$$

Proof. We will only prove the special case when n is an integer.
Recall

$$x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}).$$

So

$$(x + \Delta x)^n - x^n = (x + \Delta x - x)((x + \Delta x)^{n-1} + (x + \Delta x)^{n-2}x + \cdots + (x + \Delta x)x^{n-2} + x^{n-1}).$$

We have

$$\begin{aligned} (x^n)' = \lim_{\Delta x \to 0} \frac{(x + \Delta x)^n - x^n}{\Delta x} &= \lim_{\Delta x \to 0}((x + \Delta x)^{n-1} + (x + \Delta x)^{n-2}x + \cdots + (x + \Delta x)x^{n-2} + x^{n-1}) \\ &= x^{n-1} + x^{n-2}x + \cdots xx^{n-2} + x^{n-1} = nx^{n-1}. \end{aligned}$$

□

Remark. Alternatively, combine the fact that $x' = 1$ and the Leibniz rule.

Example 4.2.1.

$$\begin{aligned} (x^3)' &= 3x^2, && x \in \mathbb{R} \\ (\sqrt{x})' &= \frac{1}{2}x^{-\frac{1}{2}}, && x > 0. \quad \text{Caution: } x \text{ can not be } 0. \\ (\sqrt[3]{x})' &= \frac{1}{3}x^{-\frac{2}{3}}, && x \neq 0. \quad \text{Caution: } x \text{ can be negative.} \\ (x^{\frac{3}{2}})' &= \tfrac{3}{2}x^{\frac{1}{2}}, && x > 0. \end{aligned}$$

Theorem 5 (Exponential functions and Logarithmic functions)**.**

$$\boxed{(e^x)' = e^x; \quad (a^x)' = a^x \ln a,} \qquad a > 0, a \neq 1, x \in \mathbb{R}.$$

$$\boxed{(\ln x)' = \frac{1}{x}; \quad (\log_a x)' = \frac{1}{x \ln a},} \qquad a > 0, a \neq 1, x > 0.$$

general case can be derived from base e case via the chain rule.

$\ln a - \ln b = \ln\left(\frac{a}{b}\right)$

$\ln\left(\frac{x+\Delta x}{x}\right)$

$a \ln x = \ln(x^a)$

Proof. (Optional!)

$$(\ln x)' = \frac{1}{x} \iff \lim_{\Delta x \to 0} \frac{\ln(x+\Delta x) - \ln x}{\Delta x} = \frac{1}{x}$$

$$\iff \lim_{\Delta x \to 0} \frac{\ln(1+\frac{\Delta x}{x})}{\frac{\Delta x}{x}} = 1$$

$\frac{\ln(1+y)}{y}$

$$\iff \lim_{y \to 0} \ln(1+y)^{\frac{1}{y}} = 1, \quad \text{(change variable: } y := \tfrac{\Delta x}{x}\text{)}$$

$x > 0$

when $\Delta x \to 0$

$y = \frac{\Delta x}{x} \to 0$

$$\iff \lim_{y \to 0} (1+y)^{\frac{1}{y}} = e$$

(alternative definition of e)

$$\iff \lim_{z \to +\infty} \left(1+\frac{1}{z}\right)^z = \lim_{y \to 0^+} (1+y)^{\frac{1}{y}} = e \quad \text{(change variable:} z = \frac{1}{y}\text{)}$$

when $y \to 0^+$
$z \to +\infty$

and $\lim_{z \to -\infty} (1+\frac{1}{z})^z = \lim_{y \to 0^-} (1+y)^{\frac{1}{y}} = e$ (definition of e).

when $y \to 0^-$
$z \to -\infty$

$$(e^x)' = e^x \iff \lim_{\Delta x \to 0} \frac{e^{x+\Delta x} - e^x}{\Delta x} = e^x$$

$$\iff \lim_{\Delta x \to 0} \frac{e^{\Delta x} - 1}{\Delta x} = 1$$

$$\iff \lim_{y \to 0} \frac{y}{\ln(1+y)} = 1, \quad \text{(let } y = e^{\Delta x} - 1\text{)}$$

$$\iff \lim_{y \to 0} \frac{\ln(1+y)}{y} = \left.\frac{d \ln x}{dx}\right|_{x=1} = 1.$$

For general a: The formulae can be deduced from the preceding special case of $a = e$ using the chain rule (Section 4.3). □

Remark. 1. Instead of the definition given in Section 2.5, some books use $\lim_{y \to 0}(1+y)^{\frac{1}{y}}$ as the definition of e.

2. The formula for $(e^x)'$ and the formula for $(\ln x)'$ imply each other, as e^x and $\ln x$ are "inverse functions" of each other. (Cf. Chapter 5.)

Example 4.2.2.

1. $(\sqrt{x} + 2^x - 3\log_2 x)' = (\sqrt{x})' + (2^x)' - 3(\log_2 x)' = \frac{1}{2}x^{-\frac{1}{2}} + 2^x \ln 2 - \frac{3}{x \ln 2}$